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# THE NUMERICAL CALCULATION OF TWO-DIMENSIONAL EFFECTIVE MODULI FOR MICROCRACKED SOLIDS

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**Abstract**—There exist several micromechanics models for the determination of the effective moduli of microcracked solids, and crack density is the only parameter in these models that characterizes the effect of microcracking. A numerical hybrid BEM method, in conjunction with a unit cell model, is proposed in the present paper to evaluate these micromechanics models. A unit cell, which can be considered as a representative block in the solid, contains randomly distributed microcracks. The unit cell is then assumed to be periodic in the solid so as to account for interactions between cracks inside and outside the cell. There are stochastic variations of the estimated moduli for different microcrack distributions. Two groups of microcracks with the same crack density, one with a low number of large cracks and the other with a large number of small cracks, show the same range of stochastic variations and the same mean of effective moduli for random distributions of microcracks. The effective moduli based on this numerical method for randomly distributed cracks and parallel cracks are compared with those from various micromechanics models. While the differential method provides the closest estimation to the mean of the numerical results at low crack density, the generalized self-consistent method is much more accurate at relatively high crack density.

## 1. INTRODUCTION

There are several micromechanics models to estimate the effective elastic-moduli of solids containing microcracks, such as the dilute or non-interacting solution [e.g., Kachanov (1992)], the self-consistent method (Budiansky and O'Connell 1976), the Mori–Tanaka method [e.g., Benveniste (1987)], the differential method [e.g., Norris (1985), Zimmerman (1991)], and the generalized self-consistent method (Huang *et al.* 1994). The differences among these models stem from the way they account for interactions among microcracks. For example, the interaction is neglected in the dilute or non-interacting solution and is partially accounted for in the other models. A single parameter, crack density, is adopted in all these models to characterize the effect of microcracking. However, there are few experimental data with which to verify these models since it is difficult to measure the crack density of a microcracked solid. Vavakin and Salganik (1975) conducted experiments on a thin elastic sheet containing an array of randomly oriented slits. Kachanov (1994) pointed out that the centers of these cracks were arranged on a square lattice, but were not randomly located. Zimmerman (1985) examined several sets of experimental data on cracked ceramics and showed that they were all consistent with the various micromechanics models, insofar as the relationship between the effective Young's modulus and shear modulus is concerned.

Kachanov (1992) proposed a numerical method using a two-dimensional analysis to verify these models for the case of two-dimensional analyses, i.e., all microcracks parallel to the $x_3$ axis. Twenty-five microcracks with the same length were randomly generated in locations and orientations within a square in an infinite matrix. The crack density was evaluated from the crack length and the area of the square, and the effective moduli were obtained from the crack opening and sliding displacements in the square. This approximate solution showed that the mean of the effective moduli for these randomly generated microcracks is closest to the estimate given by the dilute or non-interacting solution, which was justified by the cancellation of microcrack shielding and anti-shielding.

Huang *et al.* (1994) pointed out that since there were no microcracks embedded outside the square in Kachanov's (1992) analysis, the interaction between microcracks inside and outside the square was neglected. Therefore, more microcracks were embedded in a layer outside the square, while maintaining the same crack density. The layer thickness was chosen such that any microcracks outside the layer would not significantly affect the estimation of moduli based on the square. Their solution showed that the mean of the effective moduli for randomly generated cracks is closer to the estimate from the generalized self-consistent method than that from the dilute or non-interacting solution.

The aim of the present study is to provide a different numerical method that can deal with the problem using 25 microcracks inside the square and an infinite number of microcracks outside the square so that the interactions between microcracks inside and outside the square can be accounted for accurately. The numerical results for randomly generated microcracks are compared with the micromechanics models in order to determine which micromechanics model provides the optimum estimation of the moduli. The following sections are limited to two-dimensional, plane-strain analysis.

## 2. MICROMECHANICS MODELS

Budiansky and O'Connell (1976) defined the crack density (2D) for a microcracked solid as

$$\varepsilon = N\langle a^2\rangle \tag{1}$$

where $N$ is the number of microcracks per unit area in the plane, $a$ is the half length of a microcrack, and $\langle a^2\rangle$ stands for the average of $a^2$ over all microcracks. The matrix model is assumed to be isotropic. Two types of microcrack distributions are considered in the following: randomly distributed cracks and parallel cracks.

### 2.1. *Randomly distributed cracks*

Microcracks are assumed to be completely randomly distributed in size, orientation, and location in a solid (Fig. 1). The microcracked solid has in-plane isotropy and is characterized by the in-plane stress–strain relation

$$\varepsilon_{\alpha\beta} = \frac{1}{2\bar{G}}\sigma_{\alpha\beta} - \tfrac{1}{4}\left(\frac{1}{\bar{G}} - \frac{1}{\bar{B}}\right)\sigma_{\gamma\gamma}\delta_{\alpha\beta} \quad \alpha, \beta = 1, 2 \tag{2}$$

where $\sigma_{\gamma\gamma} = \sigma_{11}+\sigma_{22}$, $\bar{G}$ is the in-plane shear modulus, and $\bar{B}$ is the in-plane bulk modulus since $\varepsilon_{11}+\varepsilon_{22} = (\sigma_{11}+\sigma_{22})/(2\bar{B})$; $\bar{G}$ and $\bar{B}$ are to be determined.

*Dilute or non-interacting solution and Mori–Tanaka method.* The interactions among microcracks are neglected in the dilute or non-interacting solution. A closed-form solution exists [e.g., Kachanov (1992)]

$$\frac{\bar{B}_{\text{dilute}}}{B} = 1\bigg/\left(1+\frac{1-\nu}{1-2\nu}\pi\varepsilon\right) \tag{3}$$

$$\frac{\bar{G}_{\text{dilute}}}{G} = \frac{1}{1+(1-\nu)\pi\varepsilon} \tag{4}$$

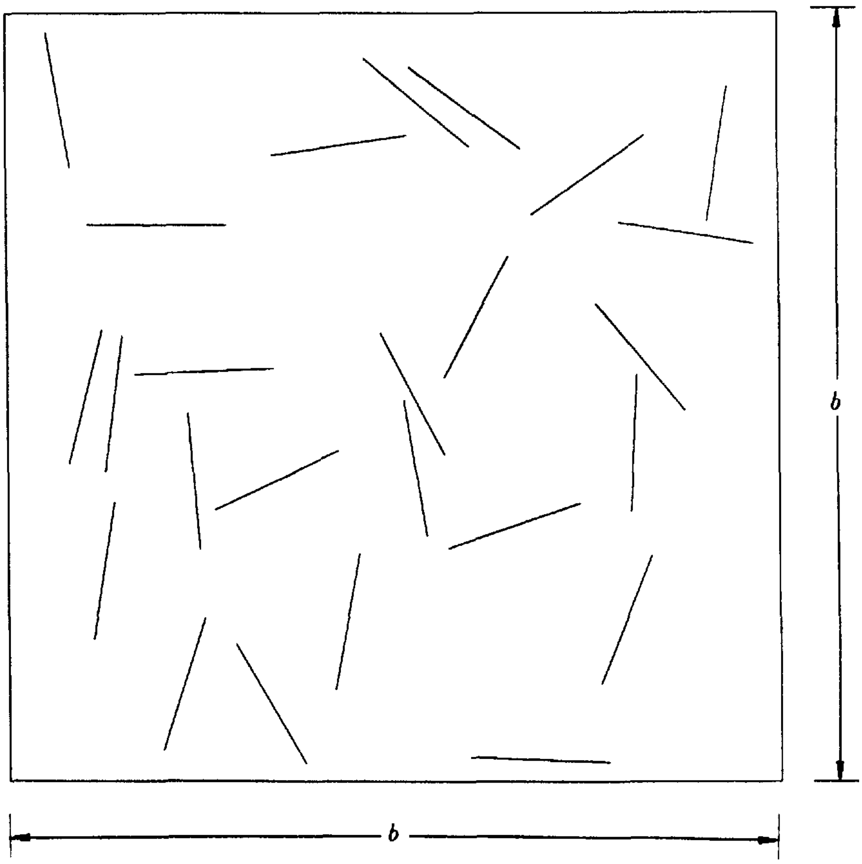


Fig. 1. Randomly distributed cracks.

where $B = E/[2(1+\nu)(1-2\nu)]$ and $G = E/[2(1+\nu)]$ are the corresponding moduli of the matrix material, i.e., the solid without microcracks; $E$ and $\nu$ are the Young's modulus and Poisson's ratio of the matrix material. The Mori–Tanaka method [e.g., Benveniste (1987)] gives results identical to those in eqns (3) and (4).

*Self-consistent method.* The self-consistent method accounts for microcrack interaction by embedding each crack directly in the effective medium—the microcracked solid (Budiansky and O'Connell 1976). A closed-form solution exists (Hu and Huang 1993):

$$\frac{\bar{B}_{\text{scm}}}{B} = \frac{(1-2\nu)(1-\pi\varepsilon)}{1-2\nu+\nu\pi\varepsilon} \tag{5}$$

$$\frac{\bar{G}_{\text{scm}}}{G} = \frac{1-\pi\varepsilon}{1-\nu\pi\varepsilon}. \tag{6}$$

*Differential method.* The differential method is an incremental form of the self-consistent method [e.g., Norris (1985), Zimmerman (1991)], where the effective moduli are governed by the coupled ordinary differential equations. A closed-form solution exists (Huang 1993):

$$\frac{\bar{B}_{\text{differential}}}{B} = \frac{1-2\nu}{e^{\pi\varepsilon}(1-\nu)-\nu} \tag{7}$$

$$\frac{\bar{G}_{\text{differential}}}{G} = \frac{1}{e^{\pi\varepsilon}(1-\nu)+\nu}. \tag{8}$$

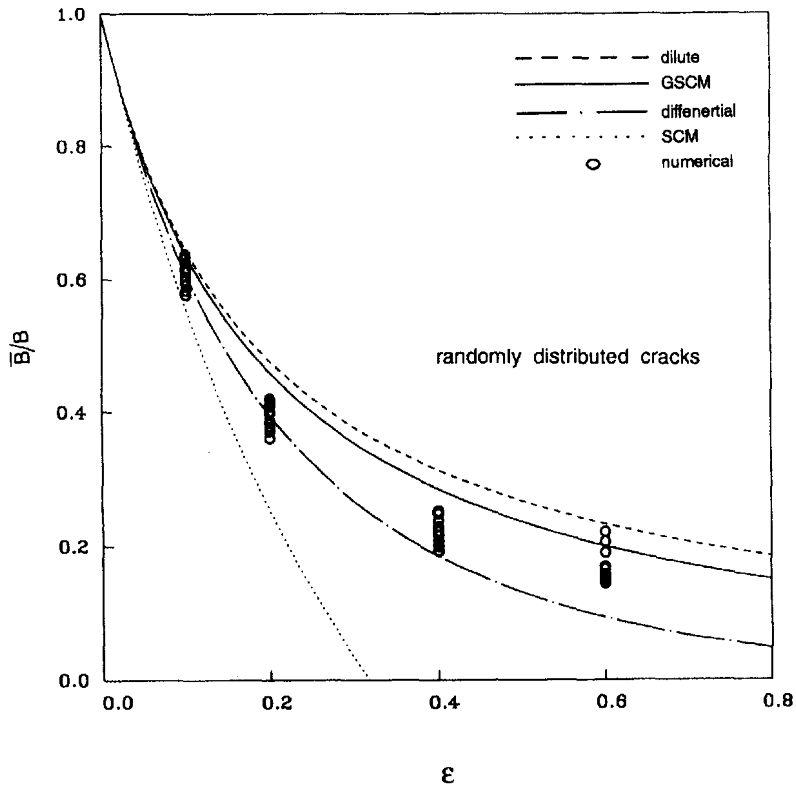


Fig. 2. Normalized in-plane bulk modulus, $\bar{B}/B$, vs crack density, $\varepsilon$, for randomly distributed cracks, estimated by the unit cell model and various micromechanics models.

*Generalized self-consistent method.* The generalized self-consistent method is a modification of the self-consistent method, where each microcrack is embedded in a surrounding matrix layer, which, in turn, is embedded in the effective medium (Huang *et al.* 1994). An *approximate* solution is found as

$$\frac{\bar{B}_{\text{gscm}}}{B} = 1\Bigg/\left(1+\frac{1-\nu}{1-2\nu}\pi\varepsilon+D_{\text{B}}(\nu)\varepsilon^2\right) \tag{9}$$

$$\frac{\bar{G}_{\text{gscm}}}{G} = \frac{1}{1+(1-\nu)\pi\varepsilon+D_{\text{G}}(\nu)\varepsilon^2} \tag{10}$$

where $D_{\text{B}}(\nu)$ and $D_{\text{G}}(\nu)$ depend on Poisson's ratio, $\nu$:

$$D_{\text{B}}(0.2) = 1.57, \quad D_{\text{B}}(0.3) = 1.97, \quad D_{\text{B}}(0.4) = 3.07 \tag{11a,b,c}$$

$$D_{\text{G}}(0.2) = 0.93, \quad D_{\text{G}}(0.3) = 0.78, \quad D_{\text{G}}(0.4) = 0.61. \tag{11d,e,f}$$

The normalized in-plane bulk and shear moduli, $\bar{B}$ and $\bar{G}$, for these methods given in eqns (3)–(11) are shown in Figs 2 and 3, respectively, with $\nu = 0.3$.

## 2.2. *Parallel cracks*

As shown in Fig. 4, microcracks are assumed to be randomly distributed in size and location, but are parallel to the $x_1$ direction. The microcracked solid is orthotropic within the plane and is characterized by the following stress–strain relations:

$$\varepsilon_{11} = A_{11}\sigma_{11}+A_{12}\sigma_{22} \tag{12a}$$

$$\varepsilon_{22} = A_{12}\sigma_{11}+A_{22}\sigma_{22} \tag{12b}$$

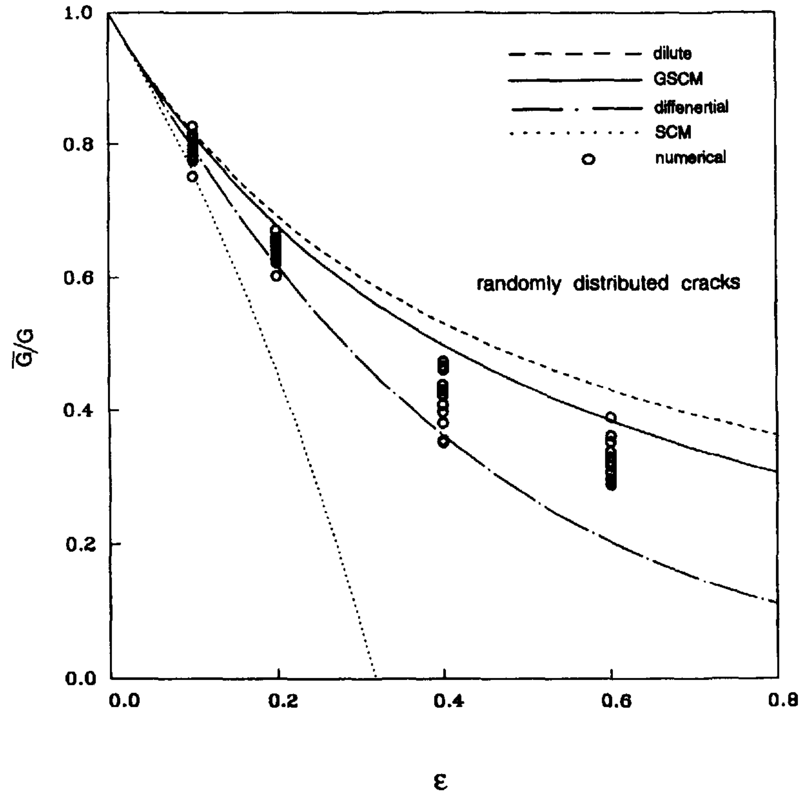


Fig. 3. Normalized shear modulus, $\bar{G}/G$, vs crack density, ε, for randomly distributed cracks, estimated by the unit cell model and various micromechanics models.

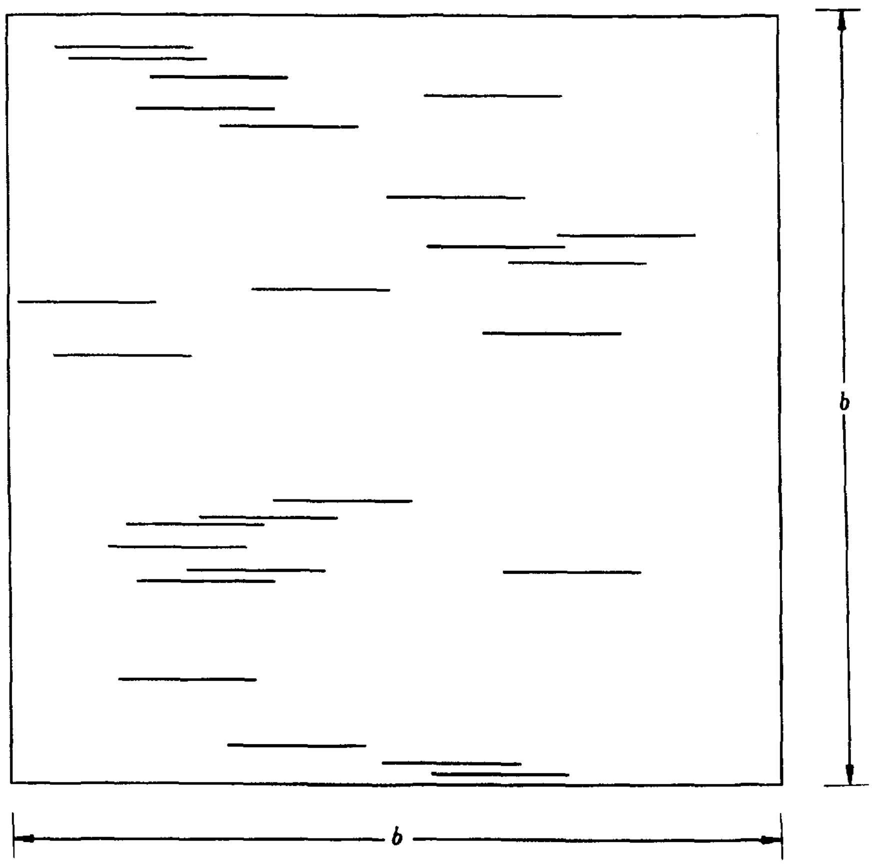


Fig. 4. Parallel cracks.

$$\varepsilon_{12} = \tfrac{1}{2}A_{66}\sigma_{12} \tag{12c}$$

where $A_{ij}$ are elastic compliances given by

$$A_{11} = \frac{1-v^2}{E}, \quad A_{12} = -\frac{v(1+v)}{E}, \quad A_{22} = \frac{1}{\bar{E}}, \quad A_{66} = \frac{1}{\bar{G}}. \tag{13}$$

Here, $E$ and $v$ are the Young's modulus and Poisson's ratio of the matrix material, and $\bar{E}$ and $\bar{G}$ are, respectively, the effective plane-strain modulus normal to the crack direction and the in-plane shear modulus of the microcracked solid.

*Dilute or non-interacting solution and Mori–Tanaka method.* A closed-form solution exists for the dilute or non-interacting solution [e.g., Kachanov (1992)]. The Mori–Tanaka method becomes identical to the non-interacting solution for microcracked solids [e.g., Benveniste (1987)]. The normalized moduli are given by

$$\frac{\bar{E}_{\text{dilute}}}{E/(1-v^2)} = \frac{1}{1+2\pi\varepsilon} \tag{14}$$

$$\frac{\bar{G}_{\text{dilute}}}{G} = \frac{1}{1+(1-v)\pi\varepsilon}. \tag{15}$$

*Self-consistent method.* Hoenig (1979) and Hu and Huang (1993) studied the effective moduli of solids containing parallel cracks using the self-consistent method. The moduli $\bar{E}$ and $\bar{G}$ are governed by the coupled nonlinear algebraic equations:

$$\frac{\bar{E}}{E/(1-v^2)} + \pi\sqrt{\frac{\bar{E}}{E/(1-v^2)}}\,C\varepsilon = 1 \tag{16}$$

$$\frac{\bar{G}}{G}\left(1+\pi\frac{1-v}{2}C\varepsilon\right) = 1 \tag{17}$$

where

$$C = \left[2\sqrt{\frac{E/(1-v^2)}{\bar{E}}} + 2 + \frac{2}{1-v}\left(\frac{G}{\bar{G}}-1\right)\right]^{1/2}. \tag{18}$$

*Differential method.* The differential method is an incremental form of the self-consistent method. The effective moduli are governed by the coupled ordinary differential equations. A closed-form solution exists (Huang 1993):

$$\frac{\bar{E}}{E/(1-v^2)} = e^{-2\pi\varepsilon} \tag{19}$$

$$\frac{\bar{G}}{G} = 1\bigg/\left(1+\frac{1-v}{2}(e^{2\pi\varepsilon}-1)\right). \tag{20}$$

To the best of our knowledge, there are no estimations for the moduli of a solid containing parallel cracks using the generalized self-consistent method. The plane-strain

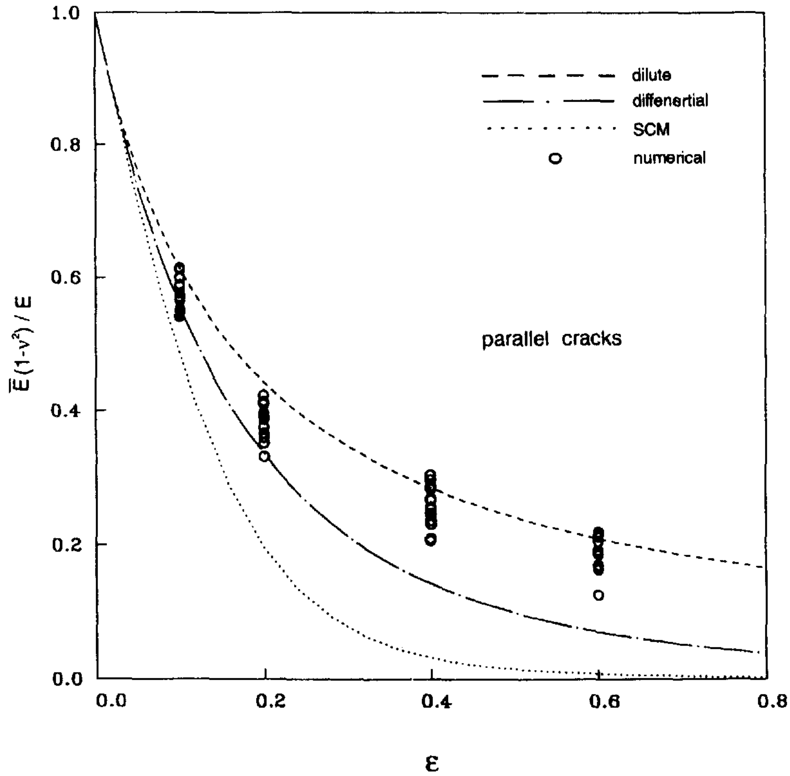


Fig. 5. Normalized plane-strain tensile modulus, $\bar{E}/[E/(1-\nu^2)]$, vs crack density, $\varepsilon$, for parallel cracks, estimated by the unit cell model and various micromechanics models.

modulus and shear modulus, $\bar{E}$ and $\bar{G}$, are shown in Figs 5 and 6, respectively, for the methods in eqns (15)–(20), with $\nu = 0.3$.

## 3. NUMERICAL METHOD

### 3.1. *Effective moduli of a microcracked solid*

Kachanov (1992) provided a general expression for the average strain in a microcracked solid:

$$\varepsilon_{\alpha\beta} = \frac{1+\nu}{E}\sigma^0_{\alpha\beta} - \frac{\nu(1+\nu)}{E}\sigma^0_{\gamma\gamma}\delta_{\alpha\beta} + \frac{1}{A}\sum_i (\langle b^{(i)}_\alpha\rangle n^{(i)}_\beta + \langle b^{(i)}_\beta\rangle n^{(i)}_\alpha)a^{(i)} \quad \alpha,\beta = 1,2 \tag{21}$$

where $\boldsymbol{\sigma}^0$ is the stress imposed on the microcracked solid; $\sigma^0_{\gamma\gamma} = \sigma^0_{11} + \sigma^0_{22}$; $E$ and $\nu$ are the Young's modulus and Poisson's ratio of the matrix material, respectively; $A$ is the in-plane area of the solid; the superscript $(i)$ denotes the $i$th crack in the solid; the summation is over all cracks; $\mathbf{b}^{(i)} = \mathbf{u}^{(i)+} - \mathbf{u}^{(i)-}$ is the displacement discontinuity vector across the top (+) and bottom (−) surfaces of the $i$th crack due to $\boldsymbol{\sigma}^0$; $\langle \mathbf{b}^{(i)} \rangle$ stands for the average of $\mathbf{b}^{(i)}$ over the crack surface; $\mathbf{n}^{(i)}$ is the unit normal pointing into the solid on the top surface (+); and $a^{(i)}$ is the half length of the $i$th crack. The displacement discontinuity vector, $\mathbf{b}^{(i)}$, is linear proportional to the stresses, $\boldsymbol{\sigma}^0$, imposed. The last term in eqn (21), i.e., the summation over all crack surfaces, gives the additional strain due to microcracking. It depends on the location, orientation, and length of the microcracks, hence, there are stochastic variations of the average strains of the microcracked solid for different microcrack distributions. The moduli of the microcracked solid evaluated from the average strain in eqn (21) will also show a range of stochastic variations, and the mean of the variations will be compared with various micromechanics models.

For imposed biaxial tension, $\sigma^0_{11} = \sigma^0_{22} = \sigma^0$ and $\sigma^0_{12} = 0$, eqn (21) gives

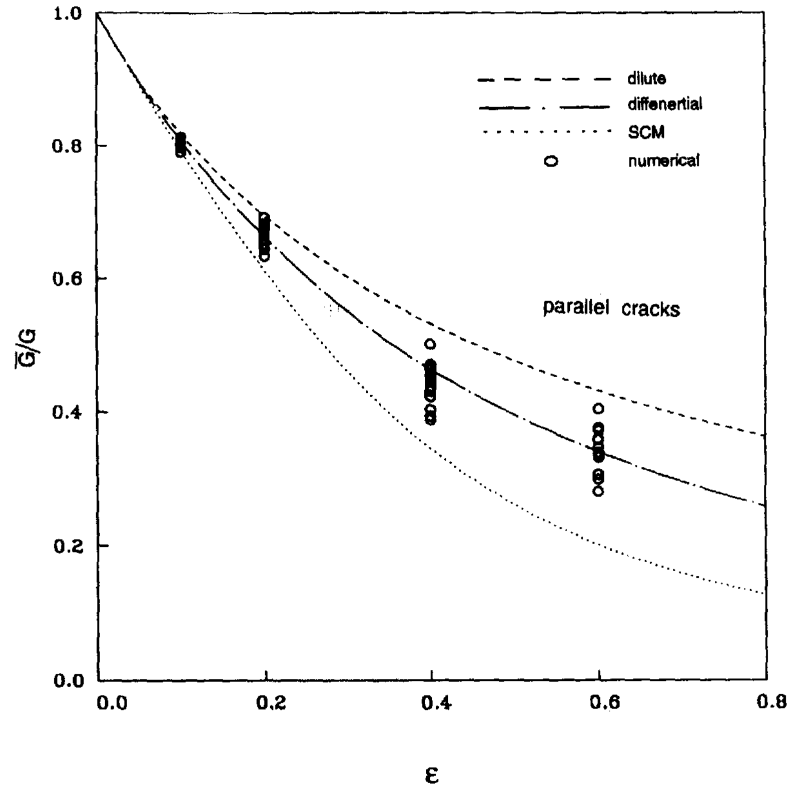


Fig. 6. Normalized shear modulus, $\bar{G}/G$, vs crack density, $\varepsilon$, for parallel cracks, estimated by the unit cell model and various micromechanics models.

$$\varepsilon_{11}+\varepsilon_{22}=\frac{\sigma^0}{B}+\frac{2}{A}\sum_i(\langle b_1^{(i)}\rangle n_1^{(i)}+\langle b_2^{(i)}\rangle n_2^{(i)})a^{(i)}. \tag{22}$$

The in-plane bulk modulus of the microcracked solid is

$$\bar{B}=\frac{\sigma^0}{\varepsilon_{11}+\varepsilon_{22}}=1\bigg/\left[\frac{1}{B}+\frac{2}{A}\sum_i(\langle b_1^{(i)}\rangle n_1^{(i)}+\langle b_2^{(i)}\rangle n_2^{(i)})\frac{a^{(i)}}{\sigma^0}\right]. \tag{23}$$

For imposed pure shear, $\sigma_{12}^0=\tau^0$ and $\sigma_{11}^0=\sigma_{22}^0=0$, eqn (21) gives

$$\varepsilon_{12}=\frac{\tau^0}{2G}+\frac{1}{A}\sum_i(\langle b_1^{(i)}\rangle n_2^{(i)}+\langle b_2^{(i)}\rangle n_1^{(i)})a^{(i)}. \tag{24}$$

The in-plane shear modulus of the microcracked solid is

$$\bar{G}=\frac{\tau^0}{2\varepsilon_{12}}=1\bigg/\left[\frac{1}{G}+\frac{2}{A}\sum_i(\langle b_1^{(i)}\rangle n_2^{(i)}+\langle b_2^{(i)}\rangle n_1^{(i)})\frac{a^{(i)}}{\tau^0}\right]. \tag{25}$$

For imposed tension in the $x_2$ direction, $\sigma_{22}^0=s^0$ and $\sigma_{11}^0=\sigma_{12}^0=0$, eqn (21) gives

$$\varepsilon_{22}=\frac{1-\nu^2}{E}s^0+\frac{2}{A}\sum_i\langle b_2^{(i)}\rangle n_2^{(i)}a^{(i)}. \tag{26}$$

The plane-strain modulus in direction 2 is

$$\bar{E}_{22} = \frac{s^0}{\varepsilon_{22}} = 1 \Bigg/ \left( \frac{1-v^2}{E} + \frac{2}{A} \sum_i \langle b_2^{(i)} \rangle n_2^{(i)} \frac{a^{(i)}}{s^0} \right). \tag{27}$$

### 3.2. *Unit cell model*

In order to estimate the elastic moduli given in eqns (23), (25) and (27) for randomly generated cracks, a representative block, $A$, must be selected. Moreover, as pointed out by Huang *et al.* (1994), the interaction between microcracks inside and outside the representative block should be accounted for. Since it is impossible to handle the infinite number of microcracks in a solid, a unit cell model is adopted in the following. Microcracks are randomly generated within a square unit cell, such as those shown in Fig. 1 for randomly distributed cracks and in Fig. 4 for parallel cracks. This unit cell can be considered as the representative block in a solid. The unit cell with the crack distribution randomly generated in the cell is assumed to repeat in the solid, i.e., to be doubly periodic in the $x_1$ and $x_2$ directions. Parallel cracks (Fig. 4) within the unit cell are randomly located, but are parallel to the $x_1$ direction. However, randomly distributed cracks are generated randomly with respect to location and orientation.

The length of the unit cell is $b$ (Fig. 7), and the origin of the $x_1$–$x_2$ coordinates coincides with the corner A. The four corners, A, B, C, and D, move to A′, B′, C′ and D′, respectively, after deformation. In order to have displacement compatibility with neighboring cells, as shown in Fig. 7, the corners A′, B′, C′, and D′ must form a parallelogram, and the relative displacements between the deformed unit cell and the parallelogram must be the same for the left and right edges and for the bottom and top edges of the unit cell. The horizontal and vertical stretches are denoted by $\Delta h$ and $\Delta v$, while the rotations of horizontal and vertical segments are denoted by $\theta_1$ and $\theta_2$, respectively.

The periodicity and compatibility of displacements on the boundary of the unit cell require

$$u_1(b, x_2) = u_1(0, x_2) + \Delta h \tag{28}$$

$$u_2(b, x_2) = u_2(0, x_2) + \theta_1 b \tag{29}$$

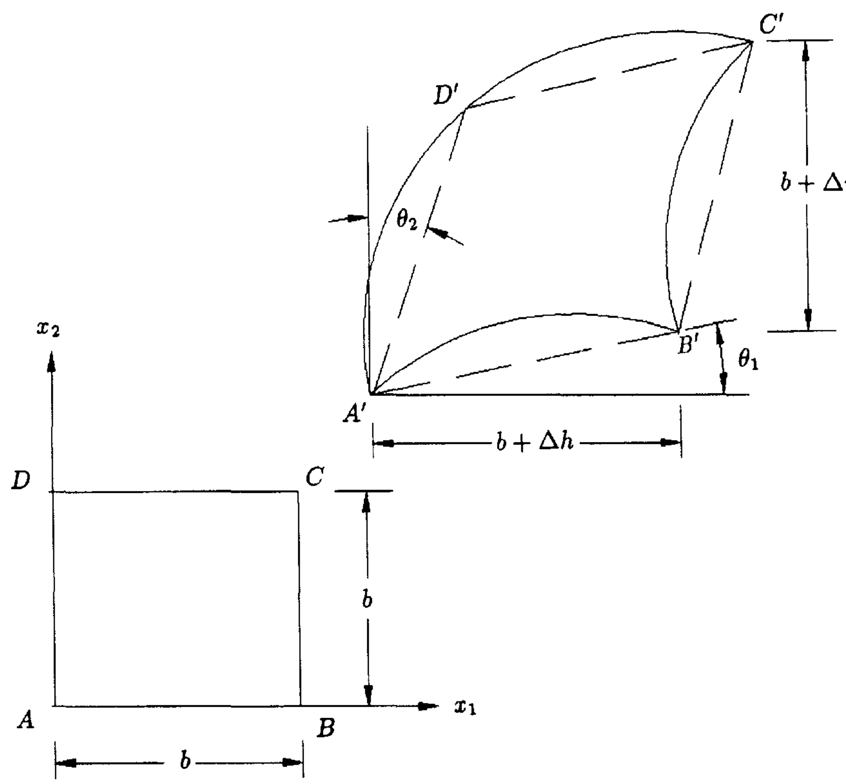


Fig. 7. Displacement compatibility on the boundary of the unit cell.

$$u_1(x_1, b) = u_1(x_1, 0) + \theta_2 b \tag{30}$$

$$u_2(x_1, b) = u_2(x_1, 0) + \Delta v \tag{31}$$

where $\Delta h$, $\Delta v$, $\theta_1$, and $\theta_2$ are to be determined. The periodicity and continuity of tractions on the boundary of the unit cell give

$$\sigma_{11}(b, x_2) = \sigma_{11}(0, x_2) \tag{32}$$

$$\sigma_{12}(b, x_2) = \sigma_{12}(0, x_2) \tag{33}$$

$$\sigma_{12}(x_1, b) = \sigma_{12}(x_1, 0) \tag{34}$$

$$\sigma_{22}(x_1, b) = \sigma_{22}(x_1, 0). \tag{35}$$

As in eqn (21), the general stresses imposed on the microcracked solid are $\sigma^0_{\alpha\beta}$ ($\alpha, \beta = 1, 2$). The periodicity and overall equilibrium of the solid requires

$$\int_0^b \sigma_{11}(0, x_2)\,\mathrm{d}x_2 = b\sigma^0_{11} \tag{36}$$

$$\int_0^b \sigma_{12}(0, x_2)\,\mathrm{d}x_2 = b\sigma^0_{12} \tag{37}$$

$$\int_0^b \sigma_{12}(x_1, 0)\,\mathrm{d}x_1 = b\sigma^0_{12} \tag{38}$$

$$\int_0^b \sigma_{22}(x_1, 0)\,\mathrm{d}x_1 = b\sigma^0_{22}. \tag{39}$$

Similarly to Kachanov (1992) and Huang *et al.* (1994), it is assumed, for simplicity, that all microcracks have the same half length, $a$. The crack density is then

$$\varepsilon = M\frac{a^2}{b^2} \tag{40}$$

where $M$ is the total number of cracks in the unit cell. The area $A = b^2$, and the summation in eqns (23), (25) and (27) is over all microcracks in the unit cell.

3.3. *Numerical method for microcracks in a finite domain*

Erdogan *et al.* (1973) developed a numerical method for multiple cracks in an *infinite* domain. Cracks are modeled as continuous distributions of dislocations. The unknown dislocation densities are governed by integral equations that can be solved by an efficient collocation method.

The boundary element method (BEM) is advantageous for problems in a solid, *finite* domain, i.e., no microcracks. The fundamental solution for the standard BEM is Kelvin's solution, a point force in an infinite matrix. The application of the reciprocal theorem leads to a boundary integral equation, which has the advantage of dealing with boundary displacements and tractions only. The evaluation of internal stresses and displacements involves more computation because it involves the integration of boundary quantities.

Chandra *et al.* (1995) combined these two methods for multiple cracks in a finite domain and developed a hybrid micro-macro BEM formulation. The modified fundamental solution for the BEM—a point force in an infinite domain containing the exact distribution of multiple cracks as in the finite domain—is solved first using the Erdogan *et al.* (1973) numerical method. The traction-free condition on crack surfaces is met in the modified fundamental solution. The application of the reciprocal theorem to the actual problem in the finite domain and the modified fundamental solution leads to a boundary integral equation on the domain's external boundary, excluding crack surfaces.

For the unit cell in the present study, the hybrid micro-macro BEM formulation (Chandra *et al.* 1995) is utilized with the boundary conditions in eqns (28)–(39). The displacements and tractions on the boundary of the unit cell are obtained numerically.

### 3.4. *Evaluation of the average displacement discontinuity vector*

In order to estimate the effective moduli of a microcracked solid, one needs the average displacement discontinuity vector, $\langle \mathbf{b}^{(i)} \rangle$, which is defined by

$$\langle \mathbf{b}^{(i)} \rangle = \frac{1}{2a^{(i)}} \int_{-a^{(i)}}^{a^{(i)}} \mathbf{b}^{(i)} \, \mathrm{d}S \quad i = 1, 2, \ldots, M \tag{41}$$

where the integration is over the surface of the *i*th crack. Since the Chandra *et al.* (1995) numerical method (discussed above) only provides the displacements on the boundary of the unit cell, an auxiliary problem is considered in the following in order to extract the integral in eqn (41).

Let an infinite matrix contain exactly the same distribution of multiple cracks as in the finite domain. The top and bottom surfaces of the *i*th crack are subject to tractions $\mathbf{t}^{(i,\mathrm{A})}$ and $-\mathbf{t}^{(i,\mathrm{A})}$ $(i = 1, 2, \ldots, M)$, respectively, where $\mathbf{t}^{(i,\mathrm{A})}$ are to be determined and the superscript (A) stands for the auxiliary problem. This problem can be solved by the Erdogen *et al.* (1973) numerical method. The tractions and displacements at the boundary of the unit cell (in this infinite matrix) are denoted by $\mathbf{t}^{(\mathrm{A})}$ and $\mathbf{u}^{(\mathrm{A})}$, respectively. The reciprocal theorem for the auxiliary problem and the unit cell problem gives

$$\sum_{i=1}^{M} \int_{i\text{th crack}} \mathbf{t}^{(i,\mathrm{A})} \cdot \mathbf{b}^{(i)} \, \mathrm{d}S = \int_{\Gamma_{\text{unit cell}}} (\mathbf{t} \cdot \mathbf{u}^{(\mathrm{A})} - \mathbf{t}^{(\mathrm{A})} \cdot \mathbf{u}) \, \mathrm{d}S \tag{42}$$

where **t** and **u** are tractions and displacements on the boundary of the unit cell, $\Gamma$unit cell, for the unit cell problem. The choice of crack surface tractions, $\mathbf{t}^{(i,\mathrm{A})}$ $(i = 1, 2, \ldots, M)$ in the auxiliary problem is now clear, i.e., select $\mathbf{t}^{(i,\mathrm{A})}$ such that $\int_{i\text{th crack}} \mathbf{t}^{(i,\mathrm{A})} \cdot \mathbf{b}^{(i)} \, \mathrm{d}S$ becomes the corresponding term in the summation in eqn (21). For example, $\mathbf{t}^{(i,\mathrm{A})} = \mathbf{n}^{(i)}$ for the estimation of in-plane bulk modulus in eqn (23), $\mathbf{t}^{(i,\mathrm{A})} = (n_2^{(i)}, n_1^{(i)})$ for shear modulus in eqn (25), and $\mathbf{t}^{(i,\mathrm{A})} = (0, n_2^{(i)})$ for plane-strain elastic modulus in the $x_2$-direction in eqn (27).

## 4. RESULTS AND DISCUSSION

As suggested by Kachanov (1992) and Huang *et al.* (1994), the number of microcracks in the cell, $M$, is fixed at 25. The crack length is related to the crack density by eqn (40). A random number generator is used to generate the locations and orientations of microcracks (only locations are randomly generated for parallel cracks). Twenty-five microcracks are generated randomly and independently in each microcrack distribution. They are regenerated if there is an intersection among cracks or an intersection between cracks and the cell boundary, because the present numerical solution procedure cannot handle the crack intersection. Fifteen microcrack distributions are generated for each crack density. The numerical method described in the previous sections is used to calculate the effective moduli for each crack distribution. The crack density, $\varepsilon$, is taken as 0.1, 0.2, 0.4, and 0.6, and the Poisson's ratio of the matrix material is fixed at 0.3.

The in-plane bulk and shear moduli, $\bar{B}$ and $\bar{G}$, for randomly generated microcracks are presented in Figs 2 and 3, respectively, along with the micromechanics models for randomly distributed cracks. The 15 generalizations of microcracks for each crack density provide stochastic variations in the moduli. The range of variations is below that for the dilute or non-interacting solution and above that for the self-consistent solution. The mean of the moduli is close to the solution using the differential method at relatively low crack density ($\varepsilon \leqslant 0.2$). As the crack density increases, the mean of the moduli becomes closer to the solution using the generalized self-consistent method. The generalized self-consistent method provides a good estimation for relatively high crack densities. It may be noted here that both differential and generalized self-consistent methods account for crack interactions.

In order to verify the numerical solution procedure and the convergence of the solution, the calculations are redone using twice as many cracks ($M = 50$) while the crack length is reduced correspondingly so as to keep the level of crack density the same (0.1, 0.2, 0.4, and 0.6). After 15 random realizations of the crack orientation and location, it is found that the ranges of stochastic variations in moduli are extremely close to those in Figs 2 and 3 and that the means of moduli for the random generalizations of microcracks are nearly identical.

The plane-strain elastic modulus and shear modulus, $\bar{E}$ and $\bar{G}$, for parallel cracks are shown in Figs 5 and 6, along with the micromechanics models. There is a range of stochastic variations in moduli, which is typically above the solution using the differential method. Although a solution using the generalized self-consistent method does not exist, it is anticipated from the trend in Figs 2 and 3 that it would be between the dilute or non-interacting solution and the differential solution (Huang and Hwang 1995) and, hence, close to the mean of moduli for randomly generated parallel cracks.

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